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Inconsistency Measurement - Technical Documentation

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1 Preliminaries

Let At be some fixed propositional signature, i. e., a (possibly infinite) set of propositions, and let $\mathcal{L}(\mathsf{At})$ be the corresponding propositional language constructed using the usual connectives $\wedge$ (and), $\vee$ (or), and $\neg$ (negation). A knowledge base $\mathcal{K}$ is a finite set of formulas $\mathcal{K} \subseteq \mathcal{L}(\mathsf{At})$. Let $\mathbb{K}$ be the set of all knowledge bases. If X is a formula or a set of formulas we write $\mathsf{At}(X)$ to denote the set of propositions appearing in X. Semantics to a propositional language is given by interpretations and an interpretation ω on At is a function $\omega : \mathsf{At} \to \{\mathsf{true}, \mathsf{false}\}$. Let $\Omega(\mathsf{At})$ denote the set of all interpretations for At. An interpretation ω satisfies (or is a model of) a proposition $a \in \mathsf{At}$, denoted by $\omega \models a$, if and only if $\omega(a) = \mathsf{true}$. The satisfaction relation $\models$ is extended to formulas in the usual way.

As an abbreviation we sometimes identify an interpretation ω with its complete conjunction, i. e., if $a_1, \ldots, a_n \in \mathsf{At}$ are those propositions that are assigned true by ω and $a_{n+1}, \ldots, a_m \in \mathsf{At}$ are those propositions that are assigned false by ω we identify ω by $a_1 \ldots a_n \overline{a_{n+1}} \ldots \overline{a_m}$ (or any permutation of this). For example, the interpretation ω_1 on $\{a, b, c\}$ with $\omega(a) = \omega(c) = \mathsf{true}$ and $\omega(b) = \mathsf{false}$ is abbreviated by $a\bar{b}c$.

For $\Phi \subseteq \mathcal{L}(\mathsf{At})$ we also define $\omega \models \Phi$ if and only if $\omega \models \phi$ for every $\phi \in \Phi$. Define furthermore the set of models $\mathsf{Mod}(X) = \{\omega \in \Omega(\mathsf{At}) \mid \omega \models X\}$ for every formula or set of formulas X. If $\mathsf{Mod}(X) = \emptyset$ we also write $X \models \perp$ and say that X is inconsistent.

2 Inconsistency Measures

Let $\mathbb{R}^{\infty}_{\geq 0}$ be the set of non-negative real values including ∞. Inconsistency measures are functions $\mathcal{I} : \mathbb{K} \to \mathbb{R}^{\infty}_{\geq 0}$ that aim at assessing the severity of the inconsistency in a knowledge base $\mathcal{K}$. The basic idea is that the larger the inconsistency in $\mathcal{K}$ the larger the value $\mathcal{I}(\mathcal{K})$ and $\mathcal{I}(\mathcal{K}) = 0$ if and only if $\mathcal{K}$ is consistent.

In the following, we give the formal definitions of currently available approaches.

Definition 1 ([Hunter and Konieczny, 2008]). The drastic inconsistency measure $\mathcal{I}_d : \mathbb{K} \to \mathbb{R}^{\infty}_{\geq 0}$ is defined as

$$\mathcal{I}_d(\mathcal{K}) = \begin{cases} 1 & \text{if } \mathcal{K} \models \perp \\ 0 & \text{otherwise} \end{cases}$$

for $\mathcal{K} \in \mathbb{K}$.

A set $M \subseteq \mathcal{K}$ is called minimal inconsistent subset (MI) of $\mathcal{K}$ if $M \models \perp$ and there is no $M' \subset M$ with $M' \models \perp$. Let $\mathsf{MI}(\mathcal{K})$ be the set of all MIs of $\mathcal{K}$.

Definition 2 ([Hunter and Konieczny, 2008]). The MI-inconsistency measure $\mathcal{I}_{\mathsf{MI}} : \mathbb{K} \to \mathbb{R}^{\infty}_{\geq 0}$ is defined as

$$\mathcal{I}_{\mathsf{MI}}(\mathcal{K}) = |\mathsf{MI}(\mathcal{K})|$$

for $\mathcal{K} \in \mathbb{K}$.

Definition 3 ([Hunter and Konieczny, 2008]). The MI^c-inconsistency measure $\mathcal{I}_{\mathsf{MI}^c} : \mathbb{K} \to \mathbb{R}^{\infty}_{\geq 0}$ is defined as

$$\mathcal{I}_{\mathsf{MI}^c}(\mathcal{K}) = \sum_{M \in \mathsf{MI}(\mathcal{K})} \frac{1}{|M|}$$

for $\mathcal{K} \in \mathbb{K}$.

For $\mathcal{K} \in \mathbb{K}$ define

$$\begin{aligned}
\mathsf{MI}^{(i)}(\mathcal{K}) &= \{M \in \mathsf{MI}(\mathcal{K}) \mid |M| = i\} \\
\mathsf{CN}^{(i)}(\mathcal{K}) &= \{C \subseteq \mathcal{K} \mid |C| = i \wedge C \not\models \perp\} \\
R_i(\mathcal{K}) &= \begin{cases} 0 & \text{if } |\mathsf{MI}^{(i)}(\mathcal{K})| + |\mathsf{CN}^{(i)}(\mathcal{K})| = 0 \\ |\mathsf{MI}^{(i)}(\mathcal{K})|/(|\mathsf{MI}^{(i)}(\mathcal{K})| + |\mathsf{CN}^{(i)}(\mathcal{K})|) & \text{otherwise} \end{cases}
\end{aligned}$$

for $i = 1, \ldots, |\mathcal{K}|$.

Definition 4 ([Mu et al., 2011]). The D_f-inconsistency measure $\mathcal{I}_{D_f} : \mathbb{K} \to \mathbb{R}^{\infty}_{\geq 0}$ is defined as

$$\mathcal{I}_{D_f}(\mathcal{K}) = 1 - \Pi_{i=1}^{|\mathcal{K}|}(1 - R_i(\mathcal{K})/i)$$

for $\mathcal{K} \in \mathbb{K}$.

Definition 5 ([Grant and Hunter, 2011]). The problematic inconsistency measure $\mathcal{I}_p : \mathbb{K} \to \mathbb{R}^{\infty}_{\geq 0}$ is defined as

$$\mathcal{I}_p(\mathcal{K}) = |\bigcup_{M \in \mathsf{MI}(\mathcal{K})} M|$$

for $\mathcal{K} \in \mathbb{K}$.

Let $\mathsf{MC}(\mathcal{K})$ be the set of maximal consistent subsets of $\mathcal{K}$, i. e.

$$\mathsf{MC}(\mathcal{K}) = \{\mathcal{K}' \subseteq \mathcal{K} \mid \mathcal{K}' \not\models \perp \wedge \forall \mathcal{K}'' \supsetneq \mathcal{K}' : \mathcal{K}'' \models \perp\}$$

Furthermore, let $\mathsf{SC}(\mathcal{K})$ be the set of self-contradictory formulas of $\mathcal{K}$, i. e.

$$\mathsf{SC}(\mathcal{K}) = \{\phi \in \mathcal{K} \mid \phi \models \perp\}$$

Definition 6 ([Grant and Hunter, 2011]). The MC-inconsistency measure $\mathcal{I}_{mc} : \mathbb{K} \to \mathbb{R}^{\infty}_{\geq 0}$ is defined as

$$\mathcal{I}_{mc}(\mathcal{K}) = |\mathsf{MC}(\mathcal{K})| + |\mathsf{SC}(\mathcal{K})| - 1$$

for $\mathcal{K} \in \mathbb{K}$.

Definition 7 ([Doder et al., 2010]). The nc-inconsistency measure $\mathcal{I}_{nc} : \mathbb{K} \to \mathbb{R}^{\infty}_{\geq 0}$ is defined as

$$\mathcal{I}_{nc}(\mathcal{K}) = |\mathcal{K}| - \max\{n \mid \forall \mathcal{K}' \subseteq \mathcal{K} : |\mathcal{K}'| = n \Rightarrow \mathcal{K}' \not\models \perp\}$$

for $\mathcal{K} \in \mathbb{K}$.

A probability function P on $\mathcal{L}(\mathsf{At})$ is a function $P : \Omega(\mathsf{At}) \to [0, 1]$ with $\sum_{\omega \in \Omega(\mathsf{At})} P(\omega) = 1$. We extend P to assign a probability to any formula $\phi \in \mathcal{L}(\mathsf{At})$ by defining

$$P(\phi) = \sum_{\omega \models \phi} P(\omega)$$

Let $\mathcal{P}(\mathsf{At})$ be the set of all those probability functions.

Definition 8 ([Knight, 2002]). The η-inconsistency measure $\mathcal{I}_\eta : \mathbb{K} \to \mathbb{R}^{\infty}_{\geq 0}$ is defined as

$$\mathcal{I}_\eta(\mathcal{K}) = 1 - \max\{\xi \mid \exists P \in \mathcal{P}(\mathsf{At}) : \forall \alpha \in \mathcal{K} : P(\alpha) \geq \xi\}$$

for $\mathcal{K} \in \mathbb{K}$.

A subset $H \subseteq \Omega(\mathsf{At})$ is called a hitting set of $\mathcal{K}$ if for every $\phi \in \mathcal{K}$ there is $\omega \in H$ with $\omega \models \phi$.

Definition 9 ([Thimm, 2016]). The hitting-set inconsistency measure $\mathcal{I}_{hs} : \mathbb{K} \to \mathbb{R}^{\infty}_{\geq 0}$ is defined as

$$\mathcal{I}_{hs}(\mathcal{K}) = \min\{|H| \mid H \text{ is a hitting set of } \mathcal{K}\} - 1$$

for $\mathcal{K} \in \mathbb{K}$ with $\min \emptyset = \infty$.

α	β	$v(\alpha \wedge \beta)$	$v(\alpha \vee \beta)$
T	T	T	T
T	B	B	T
T	F	F	T
B	T	B	T
B	B	B	B
B	F	F	B
F	T	F	T
F	B	F	B
F	F	F	F

α	$v(\neg\alpha)$
T	F
B	B
F	T

Table 1: Truth tables for propositional three-valued logic.

Definition 10 ([Xiao and Ma, 2012]). The mv inconsistency measure $\mathcal{I}_{mv} : \mathbb{K} \to \mathbb{R}^{\infty}_{\geq 0}$ is defined as

$$\mathcal{I}_{mv}(\mathcal{K}) = \frac{|\bigcup_{M \in \mathsf{MI}(\mathcal{K})} \mathsf{At}(M)|}{|\mathsf{At}(\mathcal{K})|}$$

for $\mathcal{K} \in \mathbb{K}$.

A three-valued interpretation v on At is a function $v : \mathsf{At} \to \{T, F, B\}$ where the values T and F correspond to the classical true and false, respectively. The additional truth value B stands for both and is meant to represent a conflicting truth value for a proposition. The function v is extended to arbitrary formulas as shown in Table 1. Then, an interpretation v satisfies a formula α, denoted by $v \models^3 \alpha$ if either $v(\alpha) = T$ or $v(\alpha) = B$. Then inconsistency can be measured by seeking an interpretation v that assigns B to a minimal number of propositions.

Definition 11 ([Grant and Hunter, 2011]). The contension inconsistency measure $\mathcal{I}_c : \mathbb{K} \to \mathbb{R}^{\infty}_{\geq 0}$ is defined as

$$\mathcal{I}_c(\mathcal{K}) = \min\{|v^{-1}(B)| \mid v \models^3 \mathcal{K}\}$$

for $\mathcal{K} \in \mathbb{K}$.

An interpretation distance d is a function $d : \Omega(\mathsf{At}) \times \Omega(\mathsf{At}) \to [0, \infty)$ that satisfies (let $\omega, \omega', \omega'' \in \Omega(\mathsf{At})$)

1. $d(\omega, \omega') = 0$ if and only if $\omega = \omega'$ (reflexivity),
2. $d(\omega, \omega') = d(\omega', \omega)$ (symmetry), and
3. $d(\omega, \omega'') \leq d(\omega, \omega') + d(\omega', \omega'')$ (triangle inequality).

One prominent example of such a distance is the Dalal distance d_{d} defined via

$$d_{\mathsf{d}}(\omega, \omega') = |\{a \in \mathsf{At} \mid \omega(a) \neq \omega'(a)\}|$$

for all $\omega, \omega' \in \Omega(\mathsf{At})$. If $X \subseteq \Omega(\mathsf{At})$ is a set of interpretations we define $d_{\mathsf{d}}(X, \omega) = \min_{\omega' \in X} d_{\mathsf{d}}(\omega', \omega)$ (if $X = \emptyset$ we define $d_{\mathsf{d}}(X, \omega) = \infty$).

Definition 12 ([Grant and Hunter, 2013]). The Σ-distance inconsistency measure $\mathcal{I}^{\Sigma}_{\text{dalal}} : \mathbb{K} \to \mathbb{R}^{\infty}_{\geq 0}$ is defined as

$$\mathcal{I}^{\Sigma}_{\text{dalal}}(\mathcal{K}) = \min\left\{\sum_{\alpha \in \mathcal{K}} d_{\mathsf{d}}(\mathsf{Mod}(\alpha), \omega) \mid \omega \in \Omega(\mathsf{At})\right\}$$

for $\mathcal{K} \in \mathbb{K}$.

Definition 13 ([Grant and Hunter, 2013]). The max-distance inconsistency measure $\mathcal{I}^{\max}_{\text{dalal}} : \mathbb{K} \to \mathbb{R}^{\infty}_{\geq 0}$ is defined as

$$\mathcal{I}^{\max}_{\text{dalal}}(\mathcal{K}) = \min\left\{\max_{\alpha \in \mathcal{K}} d_{\mathsf{d}}(\mathsf{Mod}(\alpha), \omega) \mid \omega \in \Omega(\mathsf{At})\right\}$$

for $\mathcal{K} \in \mathbb{K}$.

Definition 14 ([Grant and Hunter, 2013]). The hit-distance inconsistency measure $\mathcal{I}_{\text{dalal}}^{\text{hit}} : \mathbb{K} \to \mathbb{R}_{\geq 0}^{\infty}$ is defined as

$$\mathcal{I}_{\text{dalal}}^{\text{hit}}(\mathcal{K}) = \min \{|\{\alpha \in \mathcal{K} \mid d_{\mathsf{d}}(\mathsf{Mod}(\alpha), \omega) > 0\}| \mid \omega \in \Omega(\mathsf{At})\}$$

for $\mathcal{K} \in \mathbb{K}$.

A minimal proof for $\alpha \in \{x, \neg x \mid x \in \mathsf{At}\}$ in $\mathcal{K}$ is a set $\pi \subseteq \mathcal{K}$ such that

1. α appears as a literal in π
2. $\pi \models \alpha$, and
3. π is minimal wrt. set inclusion.

Let $P_m^{\mathcal{K}}(x)$ be the set of all minimal proofs of x in $\mathcal{K}$.

Definition 15 ([Jabbour and Raddaoui, 2013]). The proof-based inconsistency measure $\mathcal{I}_{P_m} : \mathbb{K} \to \mathbb{R}_{\geq 0}^{\infty}$ is defined as

$$\mathcal{I}_{P_m}(\mathcal{K}) = \sum_{a \in \mathsf{At}} |P_m^{\mathcal{K}}(a)| \cdot |P_m^{\mathcal{K}}(\neg a)|$$

for $\mathcal{K} \in \mathbb{K}$.

A set of maximal consistent subsets $\mathcal{C} \subseteq \mathsf{MC}(\mathcal{K})$ is called an MC-cover if

$$\bigcup_{C \in \mathcal{C}} C = K$$

An MC-cover $\mathcal{C}$ is normal if no proper subset of $\mathcal{C}$ is an MC-cover. A normal MC-cover is maximal if

$$\lambda(\mathcal{C}) = |\bigcap_{C \in \mathcal{C}} C|$$

is maximal for all normal MC-covers.

Definition 16 ([Ammoura et al., 2015]). The MCSC inconsistency measure $\mathcal{I}_{mcsc} : \mathbb{K} \to \mathbb{R}_{\geq 0}^{\infty}$ is defined as

$$\mathcal{I}_{mcsc}(\mathcal{K}) = |\mathcal{K}| - \lambda(\mathcal{C})$$

for all $\mathcal{K} \in \mathbb{K}$ and any maximal MC-cover $\mathcal{C}$.

A set $\{K_1, \ldots, K_n\}$ of pairwise disjoint subsets of $\mathcal{K}$ is called a conditional independent MUS partition of $\mathcal{K}$, iff each K_i is inconsistent and $\mathsf{MI}(K_1 \cup \ldots K_n)$ is the disjoint union of all $\mathsf{MI}(K_i)$.

Definition 17 ([Jabbour et al., 2014]). The CC inconsistency measure $\mathcal{I}_{CC} : \mathbb{K} \to \mathbb{R}_{\geq 0}^{\infty}$ is defined as

$$\mathcal{I}_{CC}(\mathcal{K}) = \max\{n \mid \{K_1, \ldots, K_n\} \text{ is a conditional independent MUS partition of } \mathcal{K}\}$$

for all $\mathcal{K} \in \mathbb{K}$.

An ordered set $\mathcal{P} = \{P_1, \ldots, P_n\}$ with $P_i \subseteq \mathsf{MI}(\mathcal{K})$ for $i = 1, \ldots, n$ is called an ordered CSP-partition of $\mathsf{MI}(\mathcal{K})$ if

1. $\mathsf{MI}(\mathcal{K})$ is the disjoint union of all P_i for $i = 1, \ldots, n$
2. each P_i is a conditional independent MUS partition of $\mathcal{K}$ for $i = 1, \ldots, n$
3. $|P_i| \geq |P_{i+1}|$ for $i = 1, \ldots, n-1$

Definition 18 ([Jabbour et al., 2015]). The CSP inconsistency measure $\mathcal{I}_{CSP} : \mathbb{K} \to \mathbb{R}_{\geq 0}^{\infty}$ is defined as

$$\mathcal{I}_{CSP}(\mathcal{K}) = \max\{\mathcal{W}(\mathcal{P}) \mid \mathcal{P} \in \mathcal{P}_{\mathsf{MI}(\mathcal{K})}\}$$

for all $\mathcal{K} \in \mathbb{K}$ with $\mathcal{W}(\mathcal{P}) = \sum_{i=1}^{n} w_i |P_i|$ and $\{w_n\}_{n=1}^{\infty}$ is a decreasing positive sequence with $w_1 = 1$.

In the above definition, we assume $w_i = 1/i$ fixed.

For a formula α let $\|\alpha\|_a$ denote the number of occurrences of the proposition a in α and $\alpha[a \to \phi]^i$ be the same formula as α where the ith occurrence of the proposition a is replaced by ϕ (if a occurs less times we define $\alpha[a \to \phi]^i = \alpha$).

Definition 19 ([Besnard, 2016][1]). The forgetting-based inconsistency measure $\mathcal{I}_{\text{forget}} : \mathbb{K} \to \mathbb{R}^{\infty}_{\geq 0}$ is defined as

$$\mathcal{I}_{\text{forget}}(\mathcal{K}) = \begin{cases} 0 & \text{if } \mathcal{K} \text{ is consistent} \\ \min_{a \in \mathsf{At}, i=1,\ldots,\|(\bigwedge \mathcal{K})\|_a} \{\mathcal{I}_{\text{forget}}((\bigwedge \mathcal{K})[a \to \top]^i), \mathcal{I}_{\text{forget}}((\bigwedge \mathcal{K})[a \to \perp]^i)\} + 1 & \text{otherwise} \end{cases}$$

for all $\mathcal{K} \in \mathbb{K}$.

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[1] Note that we give a slightly different but equivalent formalization.